Fast-convergent iterative scheme for filtering velocity signals and finding Kolmogorov scales

J. Mi, R. C. Deo, and G. J. Nathan
School of Mechanical Engineering, University of Adelaide, South Australia 5005, Australia
(Received 30 September 2004; published 13 June 2005)

The present fundamental knowledge of fluid turbulence has been established primarily from hot- and cold-wire measurements. Unfortunately, however, these measurements necessarily suffer from contamination by noise since no certain method has previously been available to optimally filter noise from the measured signals. This limitation has impeded our progress of understanding turbulence profoundly. We address this limitation by presenting a simple, fast-convergent iterative scheme to digitally filter signals optimally and find Kolmogorov scales definitely. The great efficacy of the scheme is demonstrated by its application to the instantaneous velocity measured in a turbulent jet.

DOI: 10.1103/PhysRevE.71.066304 PACS number(s): 47.27.−i, 47.80.+v

In his pioneering work [1] on turbulence, Kolmogorov derived, based on dimensional reasoning, the characteristic length of the finest-scale turbulent motions to be

$$\eta \equiv (\nu^3/\varepsilon)^{1/4}, \tag{1}$$

which is hence called the "Kolmogorov length scale." In Eq. (1), ν is the kinematic viscosity and ε is the average dissipation rate of the turbulence kinetic energy given by (e.g., Hinze [2])

$$\varepsilon = \nu\overline{(\partial u_i/\partial x_j + \partial u_j/\partial x_i)\partial u_i/\partial x_j} \tag{2}$$

with standard Cartesian tensor notation and summation on repeated indices, where i or $j=1$, 2, and 3 represent the streamwise, lateral, and spanwise directions, respectively.

The appropriate estimate of η is of significant importance for improving our understanding of the fine-scale turbulence. However, great difficulty occurs in directly measuring this characteristic scale. To obtain ε requires all the 12 terms of gradient correlations in Eq. (2) to be measured. This task cannot be realized by presently available experimental techniques. Accurate measurements of even one component of $\partial u_i/\partial x_j$, as is well known, requires a multisensor probe with extremely high spatial and temporal resolution to incorporate even the smallest scales of velocity fluctuations. Moreover, several cross-correlation terms in Eq. (2) simply cannot be measured directly now [3] or in the foreseeable future, let alone direct measurements of ε and η. In this context, it remains necessary to estimate η from hot-wire measurements of ε using the isotropic relation

$$\varepsilon = 15\nu\overline{(\partial u_1/\partial x_1)^2} \tag{3}$$

and also Taylor's hypothesis

$$\overline{(\partial u_1/\partial x_1)^2} = U_1^{-2}\overline{(\partial u_1/\partial t)^2}, \tag{4}$$

where U_1 is the local streamwise mean velocity. Substitution of Eq. (4) into Eq. (3) leads to

$$\varepsilon = 15\nu U_1^{-2}\overline{(\partial u_1/\partial t)^2}. \tag{5}$$

In addition, the Kolmogorov frequency is defined as

$$f_K \equiv U_1(2\pi\eta)^{-1}. \tag{6}$$

It is important to note that the nonfiltered or slightly filtered velocity signals u_{im} (the subscript m means "measured") is inevitably contaminated by high-frequency electronic noise (n), i.e.,

$$u_{im} = u_i + n. \tag{7}$$

This contamination causes both the spatial and temporal gradient variances to be overestimated, i.e.,

$$\overline{(\partial u_{im}/\partial x_j)^2} = \overline{(\partial u_i/\partial x_j)^2} + \overline{(\partial n/\partial x_j)^2}, \tag{8}$$

$$\overline{(\partial u_{im}/\partial t)^2} = \overline{(\partial u_i/\partial t)^2} + \overline{(\partial n/\partial t)^2}.$$

The extra terms $\overline{(\partial n/\partial x_j)^2}$ and $\overline{(\partial n/\partial t)^2}$ are the unwanted noise contributions. Therefore, to achieve accurate measurements of the velocity gradients, it is necessary that the raw velocity signals u_{im} be low-pass filtered at a specific cutoff frequency f_c to eliminate the effect of noise. The choice of f_c is critical. Too high and it will not remove noise contributions sufficiently, while too low and it will wipe out some content of the signal. The right choice for f_c is the Kolmogorov frequency f_K, i.e., the characteristic frequency of the smallest structures. However, f_K not only is a function of the flow but also varies with spatial locations in the flow so that it cannot be determined *a priori*. The method described in [4] to determine f_c *in situ* is complex, requiring two mechanical analog filters, a differentiator, a real-time spectrum analyzer, visual inspection, and optimization at *each* measurement location. This procedure is only realistic where the number of spatial locations or flow conditions is limited, and is prohibitive for experiments where these are large. In the absence of a simpler procedure, more arbitrary criteria are usually adopted, so that most previous measurements of ε must be contaminated by noise to some extent. The importance of this issue is also evident from [5] from which it is deduced that even slightly over filtering u_{im} at $f_c < f_K$ may cause substantial underestimate of the velocity gradients.

From the above discussion it is obvious that substantial benefit would arise from a procedure that could correctly obtain f_K without prior knowledge of η, or obtain both η and f_K solely from a nonfiltered signal of u_{1m}. The present work aims to address this issue, i.e., to develop a simple and ef-

 ©2005 The American Physical Society

fective scheme that can definitively obtain η and f_K from $u_{1m}(t)$, and thus a means by which to low-pass filter all the velocity components $u_{im}(t)$ at $f_c = f_K$, thereby minimizing the effect of noise on gradients $\partial u_i / \partial x_j$ and other derived quantities such as structure functions.

Let us first inspect the contributions of noise to the measured kinetic energy $\overline{u_1^2}$ and dissipation ε. This is illustrated using the one-dimensional spectral forms of $\overline{u_1^2}$ and ε for isotropy, which are (see [6])

$$\overline{u_1^2} = \int_0^\infty E_1(k_1) dk_1 \tag{9}$$

and

$$\varepsilon = 15\nu \int_0^\infty k_1^2 E_1(k_1) dk_1, \tag{10}$$

respectively, where E_1 is the one-dimensional spectrum function and k_1 is the wave number in the streamwise (x_1) direction. To demonstrate the influence of noise on $\overline{u_1^2}$ and ε, we consider the model spectrum function (e.g., [7,8])

$$E_1(k_1) = C_K \varepsilon^{2/3} k_1^{-5/3} \exp[-\alpha(k_1 \eta)^{3/4}] \tag{11}$$

where C_K is a "constant" determined empirically by experiments and $\alpha = \frac{3}{2} C_K$. Note that Eq. (11) has been verified by a large body of experimental high-Re data (see [9] which references relevant experiments). Using Eq. (6) and $k_1 = 2\pi f / U_1$ (Taylor hypothesis), Eq. (11) may be rewritten as

$$E_1(f) = C_K (2\pi/U_1)^{-5/3} f^{-5/3} \exp[-\alpha(f/f_K)^{3/4}]. \tag{12}$$

Suppose the noise-contaminated velocity spectrum to be $E_{1m} = E_1 + E_n$, where E_n is the "noise" contribution. Then, from Eqs. (9) and (10) and $k_1 = 2\pi f / U_1$, we can obtain

$$\overline{u_{1m}^2} = 2\pi U_1^{-1} \int_0^\infty (E_1 + E_n) df \tag{13}$$

and

$$\varepsilon_m = 120\pi^3 \nu U_1^{-3} \int_0^\infty f^2 (E_1 + E_n) df. \tag{14}$$

Using Eq. (12) with $C_K = 1.7$ and then $\alpha = 2.55$ (e.g., [10]) we illustrate in Fig. 1 the spectral density distributions of $\overline{u_{1m}^2}$ and ε_m with $E_n = 0$ and $E_n = 10^{-3}(f/f_K)^2$. This demonstrates that, when $f_c > f_K$, the contributions of the noise to $\overline{u_{1m}^2}$ and to ε_m are very different. For example, if $f_c = 10 f_K$, the ratios $(\overline{u_{1m}^2} - \overline{u_1^2})/\overline{u_1^2}$ and $(\varepsilon_m - \varepsilon)/\varepsilon$ are approximately 3.3% and 2000%, respectively. This indicates that the high-frequency noise contamination, if not properly filtered out, has an extremely large impact on ε_m while its influence on $\overline{u_{1m}^2}$, or more generally on $\overline{u_{im}^2}$, is very much less.

Next we examine the effects of noise on the measured Kolmogorov scales η_m and f_{Km}. Let us express the measured dissipation ε_m by

$$\varepsilon_m = \varepsilon + \varepsilon_n = \varepsilon(1 + \varepsilon_n/\varepsilon) = C\varepsilon \tag{15}$$

with $C = (1 + \varepsilon_n/\varepsilon) > 1$, where ε_n denotes the noise contribution. Substituting Eq. (15) into Eq. (1) leads to

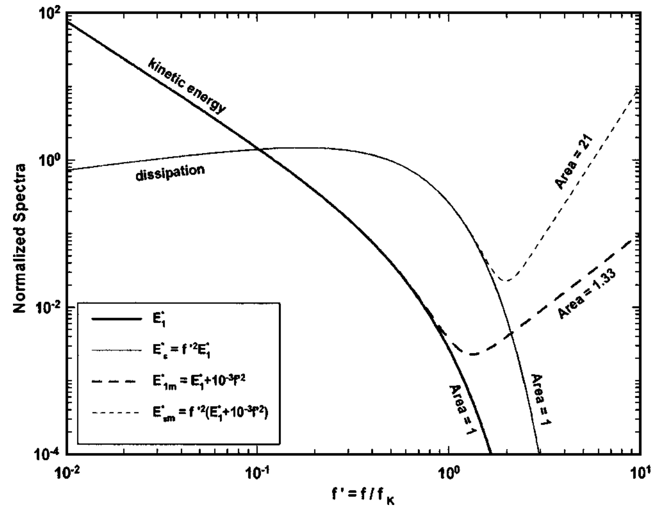

FIG. 1. (Color online) Power spectra of $\overline{u_{1m}^2}$ and ε_m with and without noise contamination.

$$\eta_m = (\nu^3 / C\varepsilon)^{1/4} = C^{-1/4} \eta. \tag{16}$$

Then, from Eq. (6), we obtain

$$f_{Km} = C^{1/4} f_K. \tag{17}$$

Accordingly, the error in ε_m, η_m, or f_{Km} from high-frequency noise can be measured by the departure of C^p from unity. As seen from Eqs. (15)–(17), the exponent p varies from $p=1$ for ε_m to $p = 1/4$ for η_m and f_{Km}. This implies a much greater error for ε_m than for η_m and f_{Km}. For example, the case $\varepsilon_m = 5\varepsilon$ results in an error of 400% (overvalued), while the corresponding error is only 33% (underestimated) for η_m and 50% (overestimated) for f_{Km}. It follows that, if we can refilter u_{im} at f_{Km} calculated from Eq. (6) and then recalculate ε_m, the noise part ε_n or $(C-1)$ [see Eq. (15)] will reduce dramatically. Subsequent recalculations of η_m, f_{Km}, and then ε_m will further reduce their errors until $\varepsilon_m \to \varepsilon$, $\eta_m \to \eta$, and $f_{Km} \to f_K$. In principle, this analysis does not require any assumptions such as isotropy or Taylor's hypothesis. That is, were the true ε or all terms of Eq. (2) measurable, it would result in such an iterative scheme by which all noise contaminations of u_{im} $(i=1,2,3)$ can be optimally filtered and therefore by which the "true" ε, η, and f_K can be found from postprocessing of sampled signals of u_{im}. However, since the direct measurement of ε is impossible at present, and in the foreseeable future, this scheme currently can be implemented only by invoking the isotropy assumption Eq. (3) and Taylor's hypothesis Eq. (4). Here we propose it in Fig. 2 and provide more details below.

The original velocity signal $U_{1m}^{(0)}(t)$ is measured by a hot-wire probe and sampled at a very high frequency (20 kHz, say). Application of analog filters is not necessary, i.e., no filtering is needed $(f_c^{(0)} = \infty)$. Otherwise, use a high cutoff frequency $f_c^{(0)}$. With the signal collected, first calculate $\varepsilon_m^{(0)}$ from Eq. (5) and then $\eta_m^{(0)}$ from Eq. (1). Next, substituting $\eta_m^{(0)}$ into Eq. (6) results in $f_{Km}^{(0)}$. If $(f_c^{(0)} - f_{Km}^{(0)})/f_c^{(0)} > \delta$, where δ is a threshold of convergence (and should be small, say, $\leq 10^{-3}$), filter $u_{1m}^{(0)}$ at $f_c^{(1)} = f_{Km}^{(0)}$ digitally using, e.g., MATLAB.

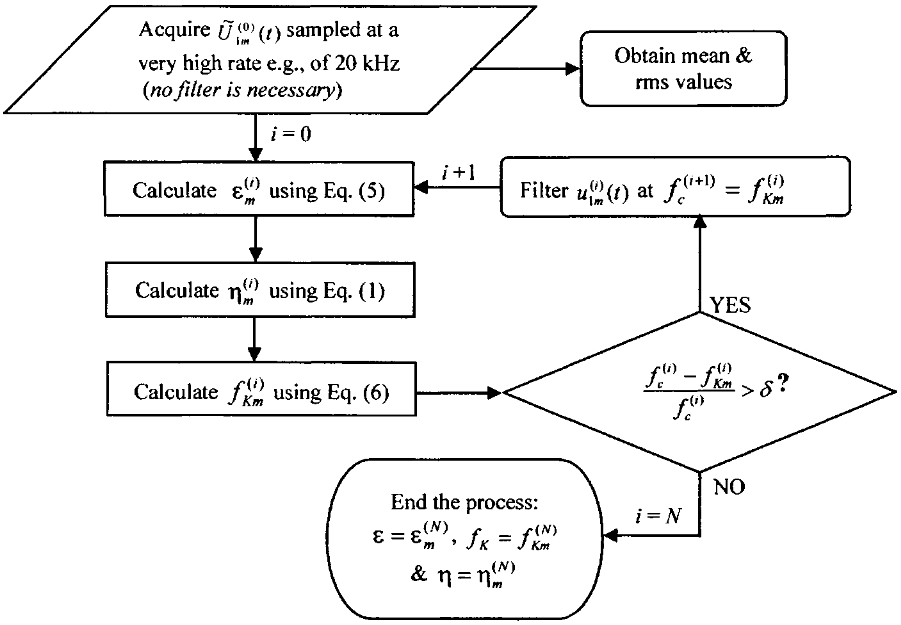

FIG. 2. Iterative scheme of digital filter to obtain η and f_K from u_{1m}.

The newly filtered velocity signal $u_{1m}^{(1)}$ is thus generated. The above process may be repeated as many times as necessary to generate $u_{1m}^{(2)}, u_{1m}^{(3)}, \ldots, u_{1m}^{(N)}$, and then $\partial u_{1m}^{(2)}/\partial t, \partial u_{1m}^{(3)}/\partial t, \ldots, \partial u_{1m}^{(N)}/\partial t$, until $(f_c^{(N)} - f_{Km}^{(N)})/f_c^{(N)} \leqslant \delta$ or until $f_{Km}^{(N)}$, $\eta_m^{(N)}$, and $\varepsilon_m^{(N)}$ have converged satisfactorily. From the converged data, we finally obtain $\varepsilon = \varepsilon_m^{(N)}$, $f_K = f_{Km}^{(N)}$, and $\eta = \eta_m^{(N)}$.

Based on Eqs. (15)–(17), both $f_{Km}^{(i)}$ and $\eta_m^{(i)}$ should converge quickly to their respective "asymptotic" values. This is true as demonstrated as an example in Table I for the case of $f_c^{(0)} = 10 f_K$, with reference to Figs. 1 and 2. Clearly, for this case, $\varepsilon_m \to \varepsilon$, $\eta_m \to \eta$, and $f_{Km} \to f_K$ just in two iterations, with an accuracy of 99.8%.

To validate the present scheme for real measurements, the instantaneous streamwise velocity $(=U_1 + u_1)$ is obtained using hot-wire anemometry along the centerline of a two-dimensional plane jet issuing from a rectangular $(w \times h = 340 \times 5.6\ \text{mm}^2)$ slot, with aspect ratio $w/h = 60$. Here only a brief description of the jet facility is given as details may be found in [11]. To ensure statistical two-dimensionality, two parallel plates $(2000 \times 1800\ \text{mm}^2)$ are attached to the short sides of the slot so that the jet mixes with ambient fluid only in the direction normal to the long sides, following, e.g., Gutmark and Wygnanski [12]. The jet exit velocity is $U_j \approx 8$ m/s, which corresponds to a Reynolds number $\text{Re} \equiv U_j h/\nu$ of approximately 3000.

Velocity measurements are performed over the region $20 \leqslant x_1/h \leqslant 160$ using a single hot-wire (tungsten) probe.

TABLE I. Use of the iterative scheme for $f_c^{(0)} = 10 f_K$.

Iterations i	$f_c^{(i)} \approx$	$\varepsilon_m^{(i)} \approx$	$\eta_m^{(i)} \approx$	$f_{Km}^{(i)} \approx$
0	$10 f_K$	17.0ε	0.5η	$2.0 f_K$
1	$2.0 f_K$	1.10ε	0.98η	$1.02 f_K$
2	$1.02 f_K$	1.02ε	0.998η	$1.002 f_K$

The hot-wire sensor is 5 μm in diameter and approximately 0.8 mm in length, aligned in the spanwise (x_3) direction. Velocity signals obtained at all the measured locations are low-pass filtered with a high and identical cutoff frequency of $f_c = 9.2$ kHz. Then they are digitized at $f_s = 2 f_c = 18.4$ kHz via a 16-channel, 12-bit analog-to-digital converter on a personal computer. The sampling duration is approximately 22 s. The wire calibration is conducted using a standard pitot tube in the jet potential core near to the exit where $\overline{u_1^2}^{1/2}/U_j \approx 0.5\%$.

The above measurements yield the original velocity signals $\tilde{U}_{1m}^{(0)}(t) = U_{1m}^{(0)} + u_{1m}^{(0)}(t)$, which are corrected for the hot-wire length of 0.8 mm using Wyngaard's approach [13], and, consequently, the original time derivatives along the jet centreline at $x_1/h \geqslant 20$ are obtained as follows:

$$\partial u_{1m}^{(0)}/\partial t \approx \Delta u_{1m}^{(0)}/\Delta t = f_s[u_{1m}^{(0)}(t + f_s^{-1}) - u_{1m}^{(0)}(t)] \quad \text{with } \Delta t = f_s^{-1}. \tag{18}$$

In this region, the original mean velocity $U_{1m}^{(0)}$ (not presented) is found to follow closely the relation $U_c/U_j \sim (x_1/h)^{-1/2}$ (here U_c denotes the centreline mean velocity), which is required by self-preservation of the jet. This relation and also that for the half-velocity width, i.e., $L_{1/2}/h \sim x_1/h$, are well satisfied by previous data (e.g., [4]) obtained in the far field.

For high-Re flows, it is usually considered that the dissipation of turbulent kinetic energy ε out of the smallest-scale structures is equal to the supply rate of the turbulence energy from the large-scale structures, which is of order U_0^3/L_0, where U_0 and L_0 are the local characteristic velocity and length scales (see, e.g., [14]). Based on this argument, we obtain $\varepsilon \sim U_c^3/L_{1/2}$, by taking $U_0 = U_c$ and $L_0 = L_{1/2}$, for a plane jet. It follows that self-preservation of the flow further requires

$$\varepsilon(hU_j^{-3}) = C_\varepsilon (x_1/h)^{-5/2} \tag{19}$$

and

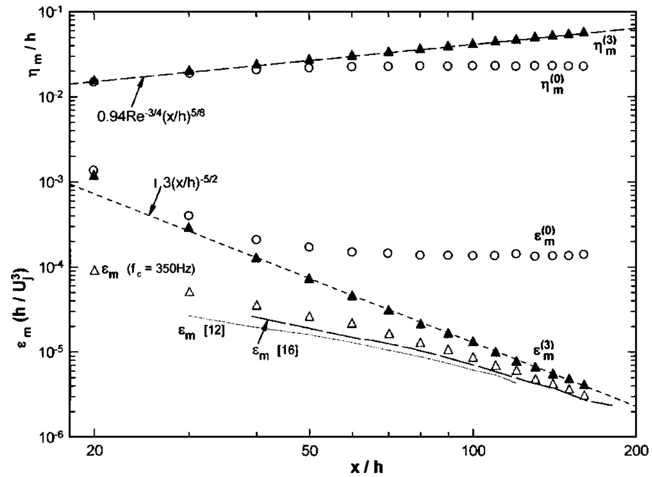

FIG. 3. (Color online) Centerline variations of $\varepsilon_m(hU_j^{-3})$ and $\eta_m h^{-1}$ for the plane jet.

$$\eta/h = C_\eta (x_1/h)^{5/8}, \tag{20}$$

where C_ε and C_η are constants determined by experiments. Indeed, both Eqs. (19) and (20) have been verified by Antonia *et al.* [4] who found that $C_\varepsilon \approx 1.3$ and $C_\eta \approx 0.94\,\mathrm{Re}_j^{-3/4}$.

These relations provide a rigorous basis against which to validate the proposed iterative filtering scheme using our jet centerline velocity signal $\tilde{U}_{1m}^{(0)}(t) = U_{1m}^{(0)} + u_{1m}^{(0)}(t)$ for $x_1/h \geqslant 20$.

Figure 3 presents the results of ε_m and η_m calculated from the original signal and after three iterations of digital filtering. The iterative procedure is involved in obtaining $\varepsilon_m^{(i)}$ from (5) via (18) and $\eta_m^{(i)}$ and $f_{Km}^{(i)}$ from (1) and (6). As with the example presented earlier, it is found that only three iterations ($N=3$) are required for $\varepsilon_m^{(i)}$, $\eta_m^{(i)}$, and $f_{Km}^{(i)}$ at all measured locations to converge to their asymptotic values when setting $\delta = 10^{-3}$ (note that a smaller value of δ requires more iterations; e.g., $N=5$ if $\delta = 10^{-4}$). Also shown in Fig. 3 are Eqs. (19) and (20) using $C_\varepsilon \approx 1.3$ and $C_\eta \approx 0.94\,\mathrm{Re}_j^{-3/4}$ obtained by Antonia *et al.* [4].

Figure 3 demonstrates that the data of $\varepsilon_m^{(3)}$ and $\eta_m^{(3)}$ agree almost identically with the curves of Eqs. (19) and (20) throughout the measured region. This provides strong support both for the validity of the iterative scheme and for the values $C_\varepsilon \approx 1.3$ and $C_\eta \approx 0.94\,\mathrm{Re}_j^{-3/4}$. It is also noted that, while $\varepsilon_m^{(0)}$ lies well above Eq. (20), $\eta_m^{(0)}$ falls below Eq. (19). As x_1 increases, f_K decreases and thus the ratio $f_c^{(0)}/f_K$ increases so that the relative contribution of electronic noise grows rapidly. Consequently, $\varepsilon_m^{(0)}$ and $\eta_m^{(0)}$ show increased departures from their true values with increased x_1. This problem may be responsible for errors in some previously reported data, since the delicate setting for the low-pass filter described in [4] may not work very well due to the fact that f_K is unknown *a priori*. A typical example of data that is apparently contaminated by noise can be found in [15] (see their Figs. 3 and 4), where various velocity and scalar structure functions are reported. That these measurements are quite recent highlights the need for a more rigorous approach to obtain such data for basic research of turbulence.

Figure 3 also illustrates the effect of overfiltering $u_{1m}(t)$. In this case, the measured values of ε_m and η_m can never agree with the relations (19) and (20). The results presented for this case are obtained from filtering all $u_{1m}^{(0)}(t)$ along the centerline at $f_c = 350$ Hz, which is one-half of f_K for $x_1/h = 160$. Interestingly, these measurements of ε_m agree quite well with those from previous studies [12,16], where their values of $\varepsilon_m h/U_j^3$ were converted from their reported data. Gutmark and Wygnanski [12] and Heskestad [16] did not offer any detailed information about their filter settings. However it is quite clear that their measurements do not follow relations (19) and (20) and therefore that their signals might well be overfiltered. Hence, their data offered wrong information that Eq. (19) does not hold in the far field of their jet where self-similarity of the mean flow has developed. The above observation provides further support for our iterative scheme.

In summary, based on the definitions of the Kolmogorov scales η and f_K, a fast-convergent iterative scheme has been developed to both optimally filter the noise-contaminated hot-wire velocity signals and simultaneously find the values of η and f_K. The scheme, implemented via use of the isotropy assumption and Taylor's hypothesis, has been validated firmly by its application to the instantaneous velocities measured in a turbulent plane jet (presented) and other flows (not presented here). The proposed scheme is believed to have wide significance for basic research on fine-scale turbulence because nearly all experimental studies in this field to date have used hot-wire measurements [17]. It has implications for past measurements, and the conclusions based on them, which may suffer to an unknown extent from the contamination of noise resulting from underfiltered or overfiltered data. Application of our scheme will allow future measurements to determine the correct cutoff frequency, as well as η and f_K, easily and unambiguously, so generating reliable data for better understanding of small-scale turbulence. Our scheme is much simpler and more rigorous than the previous scheme [4] and even obviates the need for analog filters, differentiators, and the like.

The present scheme also applies for the measurement of temperature (scalar) using cold-wire anemometer for estimates of the Batchelor, instead of the Kolmogorov, scale.

The support of the Australian Research Council is gratefully acknowledged.

[1] N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941) [Proc. R. Soc. London, Ser. A **434**, 9 (1991)].
[2] J. O. Hinze, *Turbulence*, 2nd ed. (McGraw-Hill, New York, 1975), p. 218.
[3] L. W. Browne, R. A. Antonia, and D. A. Shah, J. Fluid Mech. **179**, 307 (1987).
[4] R. A. Antonia, B. R. Satyaprakash, and A. K. M. F. Hussain, Phys. Fluids **23**, 695 (1980).
[5] J. Mi and G. J. Nathan, Exp. Fluids **34**, 687 (2003).
[6] F. H. Champagne, J. Fluid Mech. **86**, 67 (1978).
[7] S. Corrsin, Phys. Fluids **7**, 1156 (1964).
[8] Y.-H. Pao, Phys. Fluids **8**, 1063 (1965).
[9] S. G. Saddoughi and S. V. Veeravalli, J. Fluid Mech. **268**, 333 (1994).
[10] L. M. Smith and W. C. Reynolds, Phys. Fluids A **3**, 992 (1991).
[11] R. Deo, J. Mi, and G. J. Nathan (unpublished).
[12] E. Gutmark and I. Wygnanski, J. Fluid Mech. **73**, 465 (1976).
[13] J. C. Wyngaard, J. Sci. Instrum. **1**, 1105 (1968).
[14] H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, 1972).
[15] E. Lévêque, G. Ruiz-Chavarria, C. Baudet, and S. Ciliberto, Phys. Fluids **11**, 1869 (1999).
[16] G. Heskestad, J. Appl. Mech. **87**, 735 (1965).
[17] K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid Mech. **29**, 435 (1997).